

ON THE IDENTIFIABILITY OF BINARY SEGRE PRODUCTS

CRISTIANO BOCCI AND LUCA CHIANTINI

Abstract. *We prove that a product of $m > 5$ copies of \mathbb{P}^1 , embedded in the projective space \mathbb{P}^r by the standard Segre embedding, is k -identifiable (i.e. a general point of the secant variety $S^k(X)$ is contained in only one $(k+1)$ -secant k -space), for all k such that $k+1 \leq 2^{m-1}/m$.*

MSC. 14M12, 14N05, 62H17

1. INTRODUCTION

In this paper, we study secant varieties S^k of Segre products of projective spaces, with special focus on products of many copies of \mathbb{P}^1 (binary Segre products or Bernoulli models, in Algebraic Statistics). We are mainly concerned with the number of secant spaces passing through a general point of a secant variety.

In the literature, one finds several methods for computing the dimension of secant varieties of products. Let us just mention the inductive method introduced by Abo, Ottaviani and Peterson in [1], which provides a procedure for detecting when the dimension coincides with the expected one. In the specific case of products of copies of \mathbb{P}^1 , a complete description of the dimension of secant varieties has been obtained by Catalisano, Geramita and Gimigliano in [5] and [6]. When the number of copies m of \mathbb{P}^1 is bigger than 4, they prove that S^k has always the expected dimension. From our point of view, the result implies that, when the secant variety S^k does not fill the ambient space and $m > 4$, then through a general point of S^k one finds only finitely many $(k+1)$ -secant k -spaces. In this paper, we go one step further and we ask *how many* secant spaces one finds through a general point of S^k . Our main result is:

Theorem 1.1. *Let X be a product of $m > 5$ copies of \mathbb{P}^1 , embedded in the projective space \mathbb{P}^r , $r = 2^m - 1$, by the standard Segre embedding. Let $S^k(X)$ be the k -th secant variety of X , generated by $(k+1)$ -secant k -spaces. If $k+1 \leq 2^{m-1}/m$, then a general point of $S^k(X)$ is contained in only one $(k+1)$ -secant k -space.*

Following a notation suggested by applications to Algebraic Statistics, we say that a variety X is k -identifiable when through a *general* point of the secant variety $S^k(X)$, there is only one $(k+1)$ -secant k -space. (Those who would prefer "generically k -identifiable" here, should consider that there are *always* points of $S^k(X)$, e.g. points of X , for which the number jumps to infinity.) With this notation, our result can be rephrased by saying that a product of $m > 5$ copies of \mathbb{P}^1 is k -identifiable, as soon as $k+1 \leq 2^{m-1}/m$ (i.e. $m - \log_2(m) \geq \lceil \log_2(k+1) \rceil + 1$).

From this last point of view, k -identifiability has been studied because of its application to Algebraic Statistics and other fields. Using methods of Algebraic Geometry, Elmore, Hall and Neeman proved in [12] the following asymptotic result: when the number m of copies of \mathbb{P}^1 is "very large" with respect to k , then the binary Segre product is k -identifiable.

As far as we know, the best bound for identifiability of binary products has been obtained by Allman, Matias and Rhodes in [3] (Corollary 5). They prove that the product is k -identifiable when $m > 2\lceil \log_2(k+1) \rceil + 1$. Thus, they give a lower bound for 2^m which is quadratic with respect to $k+1$. Our theorem provides an extension of these results. In order to compare with the aforementioned bounds, notice that $(\mathbb{P}^1)^m$ cannot be k -identifiable for $k > 2^m/(m+1) - 1$, by a simple dimensional count, explained in Section 2. Thus, the maximal k for which identifiability makes sense is $k_{max} = \lfloor 2^m/(m+1) \rfloor - 1$. The result of Allman, Matias and Rhodes, rewritten from this point of view, says that $(\mathbb{P}^1)^m$ is k -identifiable for $k+1 \leq 2^{(m-1)/2}$. Our theorem extends this bound, for we prove that:

$$(\mathbb{P}^1)^m \text{ is } k\text{-identifiable for } k+1 \leq \frac{2^{m-1}}{m}.$$

Still more or less half way from the maximum, but a sensible improvement, anyway. For example, for $m = 10$, k_{max} is 92, our bound proves the k -identifiability for $k \leq 50$, while Allman, Matias and Rhodes give identifiability for $k \leq 16\sqrt{2}$.

Our method is strongly based on the result on the dimension of secant varieties, contained in [6]. Indeed, for a variety X , both the dimension of secant varieties and the number of secant spaces passing through a point, are linked to the existence of very degenerate subvarieties passing through $k+1$ general points of X . We explain this fact in details, through the next section. Using this remark, transferring results on the dimension of secant varieties to results on the identifiability, becomes straightforward. At the end of the paper, we will explain why we need the assumption $m > 5$. Namely, we prove that $(\mathbb{P}^1)^5$ is not 4-identifiable.

Let us finish by stating the following conjecture, suggested by our analysis.

Conjecture 1.2. For $m > 5$ and for all $k = 1, \dots, \lfloor 2^m/(m+1) \rfloor - 1$, the binary Segre product $(\mathbb{P}^1)^m$ is k -identifiable.

2. GEOMETRIC BACKGROUND

In this section, we collect some known results on secant varieties and Segre products. We refer to [9], for details and proofs. We work over the complex field and we consider the projective space $\mathbb{P}^r = \mathbb{P}_{\mathbb{C}}^r$, equipped with the tautological line bundle $\mathcal{O}_{\mathbb{P}^r}(1)$.

If Y is a subset of \mathbb{P}^r , we denote by $\langle Y \rangle$ the linear span of Y . We say that Y is *non-degenerate* if $\langle Y \rangle = \mathbb{P}^r$. A linear subspace of dimension n of \mathbb{P}^r will be called a n -*subspace* of \mathbb{P}^r .

Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non-degenerate variety of dimension m . For any non-negative integer k , the k -*secant variety* of X is the Zariski closure in \mathbb{P}^r of the union of all k -dimensional subspaces of \mathbb{P}^r that are spanned by $k+1$ independent points of X . We denote it by $S^k(X)$, or S^k , if no confusion arises. $S^k(X)$ can be seen as the closure of the image, under the second projection, of the *abstract secant variety*, i.e. the incidence subvariety $AbS^k(X) \subset X^{(k)} \times \mathbb{P}^r$,

$$AbS^k(X) = \{((P_0, \dots, P_k), P) : P \in \langle P_0, \dots, P_k \rangle, \text{ and the } P_i\text{'s are independent}\}.$$

Notice that $AbS^k(X)$ is *always* a variety of dimension $mk + m + k$. When $X \subset \mathbb{P}^r$ is reducible, the same definition of secant variety holds, except that we only consider linear spaces meeting every component of X . In particular, when X has $k+1$ components, the secant variety coincides with the *join* of the components (see [16]).

Definition 2.1. We say that X has k -th secant order μ if for a general point $P \in S^k(X)$, there are exactly μ unordered $(k+1)$ -uples P_0, \dots, P_k of points of X such that $P \in \langle P_0, \dots, P_k \rangle$. We say that X is (generically) k -*identifiable* if it has k -th secant order 1, i.e. if for a general point $P \in S^k(X)$, there is a unique unordered $(k+1)$ -uple P_0, \dots, P_k of points of X such that $P \in \langle P_0, \dots, P_k \rangle$.

Example 2.2. If X is the union of $k+1$ linearly independent subspaces of dimension m , then X has k -th secant order 1. This is an easy exercise of Linear Algebra, for $k+1=2$. For $k+1>2$, it follows by induction, by projecting from one linear component of X . Rational normal curves in \mathbb{P}^{2k+1} are the unique irreducible curves with k -th secant order 1. See e.g. Theorem 3.1 of [9].

From the definition of secant varieties, it follows that:

$$(1) \quad s^{(k)}(X) := \dim(S^k(X)) \leq \min\{r, mk + m + k\}.$$

The right hand side is called the *expected dimension* of $S^k(X)$.

Definition 2.3. We say that X is k -defective when a strict inequality holds in (1).

Remark 2.4. It is clear that X is k -identifiable when the projection $AbS^k(X) \rightarrow S^k(X)$ is birational. So X cannot be k -identifiable when $\dim(AbS^k(X)) > s^{(k)}(X)$. In particular, X is not k -identifiable when $r < mk + m + k$ or when X is k -defective.

Let $X \subset \mathbb{P}^r$ be a variety. We denote by $\text{Sing}(X)$ the Zariski-closed subset of singular points of X . Let $P \in X \setminus \text{Sing}(X)$ be a smooth point. We denote by $T_{X,P}$ the embedded tangent space to X at P , which is a m -subspace of \mathbb{P}^r . More generally, if P_0, \dots, P_k are smooth points of X , we will set

$$T_{X,P_0,\dots,P_k} = \langle \bigcup_{i=1}^n T_{X,P_i} \rangle.$$

The relations between secant varieties and tangent spaces to X are enlightened by the celebrated Terracini's Lemma:

Lemma 2.5. (see [15] or, for modern versions, [2], [11], [16]). *Given a general point $P \in S^k(X)$, lying in the subspace $\langle P_0, \dots, P_k \rangle$ spanned by $k+1$ general points on X , then the tangent space $T_{S^k(X),P}$ to $S^k(X)$ at P is the span T_{X,P_0,\dots,P_k} of the tangent spaces $T_{X,P_0}, \dots, T_{X,P_k}$.*

Using the correspondence between the abstract secant variety and $S^k(X)$, one obtains from Terracini's Lemma, a condition for the defectivity of X :

Theorem 2.6. (See [9], Theorem 2.5) *Let P_0, \dots, P_k be general points of X . If H is a general hyperplane tangent to X at P_0, \dots, P_k , we can consider the contact variety of H , i.e. the union Σ of the irreducible components of $\text{Sing}(X \cap H)$. If X is k -defective, then Σ is positive dimensional.*

The previous Theorem suggests a refinement of the notion of defective variety.

Definition 2.7. An irreducible, non-degenerate variety $X \subset \mathbb{P}^r$ such that $s^{(k)}(X) < r$ is k -weakly defective if for $P_0, \dots, P_k \in X$ general points, the general hyperplane H containing T_{X,P_0,\dots,P_k} is tangent to X along a variety $\Sigma(H)$ of positive dimension. $\Sigma(H)$ is called the $(k+1)$ -contact variety of H .

It turns out that k -defective implies k -weakly defective, but the converse is false. We refer to [7] and [9] for a discussion on the subject.

The main link between identifiability and weakly defective varieties lies in the following:

Theorem 2.8. (See [9], Corollary 2.7) *Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non-degenerate variety of dimension m . Assume $mk + m + k < r$. Then X is k -identifiable, unless it is k -weakly defective.*

Theorem 2.9. (See [9], Theorem 2.4) *Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non-degenerate variety of dimension m . Assume $mk + m + k < r$ and assume that X is k -weakly defective. Call Σ a general $(k+1)$ -th contact variety. Then, the k -th secant order of Σ is equal to the k -th secant order of X .*

Thus, a way to prove that a variety X is k -identifiable, at least when $r \neq mk + m + k$, is to prove that X is not k -weakly defective, or, if it is k -weakly defective, that the general contact variety Σ has k -th secant order 1.

The second cornerstone in our theory links k -defectivity and k -weakly defectivity with the existence of degenerate subvarieties, passing through $k+1$ general points in X . Namely, if X is k -defective or k -weakly defective, then it turns out that the general contact variety is highly degenerate.

Theorem 2.10. *(See [9], Theorem 2.4 and Theorem 2.5) Assume $mk + m + k < r$. If X is k -weakly defective, then a general contact variety Σ spans a linear space of dimension $\leq nk + n + k$, where $n = \dim(\Sigma)$. Moreover, X is k -defective if and only if Σ spans a space of dimension $< nk + n + k$.*

In conclusion, we obtain:

Corollary 2.11. *Assume $r > mk + m + k$. Assume that for all $n = 1, \dots, m-1$, there are no families of n -dimensional subvarieties of X , whose general element spans a linear space of dimension $\leq nk + n + k$ and passes through $k+1$ general points of X . Then X is not k -weakly defective. Hence it is k -identifiable.*

This is our starting point. In the next sections, we will obtain the k -identifiability of products of \mathbb{P}^1 's, for k in our range, by proving that subvarieties of Segre products $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$ passing through $k+1$ general points cannot be too degenerate, unless they are formed by a bunch of independent linear spaces. One should observe that both Corollary 2.11 and the second part of Theorem 2.8 cannot be inverted.

Example 2.12. The existence of families of degenerate subvarieties cannot guarantee that X is k -weakly defective. For instance, consider $X = \mathbb{P}^2$ embedded in \mathbb{P}^9 by the 3-Veronese embedding. Then X is not 1-weakly defective. Indeed, the general hyperplane tangent to X at two general points cuts a divisor which corresponds, in \mathbb{P}^2 , to a general cubic curve with two singular points. Such a cubic splits in the union of a conic and a line, and it is reduced. On the other hand, through 2 general points of X one finds a curve spanning a space of dimension $1 \cdot 1 + 1 + 1 = 3$. Namely, it is the twisted cubic, image of the line through the two points.

Example 2.13. When X is k -weakly defective, it can be k -identifiable as well. This may happen, by [9], Theorem 2.4, when the contact locus has k -th secant order 1. Examples of such varieties can be found in [9], Example 3.7, but they are singular. A smooth example was communicated us by G. Ottaviani. Take the Segre embedding of $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ in \mathbb{P}^{11} . Using a computer-aided procedure, one can find that the general hyperplane which is tangent to X at two points, is indeed tangent along a twisted cubic. The computation was indeed performed at two specific points of X , but notice that $\text{Aut}(X)$ acts transitively on pair of points. Thus X is 1-weakly defective. Since a twisted cubic curve has first secant order equal to 1 (Example 2.2), it turns out by [9], Theorem 2.4, that X is 1-identifiable. The 1-identifiability of X also follows from the Kruskal's identifiability criterion for the product of three projective spaces (see [14]).

As a consequence, one cannot use the inverse of the previous argument to determine the non-identifiability of a variety X , simply by studying degenerate subvarieties.

Remark 2.14. The degenerate subvarieties Σ , whose existence is guaranteed by Corollary 2.11 in any weakly defective variety, are not necessarily smooth, neither they are necessarily irreducible (although one can assume that they are reduced). On the other hand, one can use a monodromy-type argument (see e.g. [8], Proposition 3.1) in order to show that, when Σ is reducible, all the components are interchanged in a flat deformation, thus they are general members of a flat family. This is due to the generality of the points P_i 's. In particular, we may assume that the components share the same geometrical properties, also with respect to the linear series induced by the projections to the factors \mathbb{P}^1 .

3. PROOF OF THE THEOREM

Let us start with an useful Lemma of Linear Algebra:

Lemma 3.1. *Let H_0, \dots, H_k be subspaces of \mathbb{P}^s , such that the sum $H_0 + \dots + H_k$ is not direct. Let p be the dimension of the linear span of the H_i 's. Then, for a general choice of points $P_0, \dots, P_k \in \mathbb{P}^r$ $r \geq 1$, the dimension of the linear span of the spaces $H_i \times \{P_i\} \subset \mathbb{P}^s \times \mathbb{P}^r$ is at least $p + 1$.*

Proof. We may assume that H_0 meets the span $L = \langle H_1 \cup \dots \cup H_k \rangle$. If $d = \dim(H_0)$ and $e = \dim(L)$, then by assumption $\langle H_0 \cup L \rangle$ has dimension at most $d + e$, while for a general choice of the points P_0, P_1 ,

$$\dim(\langle (H_0 \times P_0) \cup (L \times P_1) \rangle) = d + e + 1,$$

for the two spaces belong to linearly independent copies of \mathbb{P}^s in the product. Now, the claim follows by specializing all P_2, \dots, P_k to P_1 . \square

The proof of our Main Theorem now follows soon by the main result of [6] and by the following general observation.

Lemma 3.2. *Let $Y \subset \mathbb{P}^s$ be a non-degenerate variety of dimension d which is not k -defective ($k \geq 1$). Assume $kd + k + d < s$. Then $X = Y \times \mathbb{P}^q$ ($q \geq 1$) is k -identifiable.*

Proof. If X is k -weakly defective, then by Theorem 2.10 the $(k + 1)$ -contact locus is a subvariety W of some dimension $n > 0$, contained in \mathbb{P}^{nk+n+k} , which passes through $k + 1$ general points of X . Assume that such a variety exists. Call $W' \subset Y$ the image of W in the projection $X \rightarrow Y$ and call $n' = \dim(W')$. Since W' passes through $k + 1$ general points of Y , and Y is not k -defective, then the span of W' has dimension at least $n'k + n' + k$, by the second part of Theorem 2.10. Now, notice that the fibers of the projection $W \rightarrow W'$ span a space of dimension at least $n - n'$ in \mathbb{P}^q . It follows, by Linear Algebra, that W spans a space of dimension at least $(n - n' + 1)(n'k + n' + k)$.

Now, we have to study several cases. Assume $0 < n' < n$. Then $n'(n - n') \geq n - n'$, so that $(n - n' + 1)n' \geq n$. Moreover $(n - n' + 1)k > k$. It follows that $(n - n' + 1)(n'k + n' + k)$ is bigger than $nk + n + k$, so we get a contradiction.

Assume $n = n' > 0$. By construction, the linear series L which sends W to \mathbb{P}^s , passing through the embedding $W \subset X$ and the projection $X \rightarrow Y$, has dimension equal to the span of W' . Hence it has dimension $kn + n + k$, in our case. Call L' the linear series defining the projection $W \rightarrow \mathbb{P}^q$. If the image of W in \mathbb{P}^q has dimension at least 1, then the embedding of W in $\mathbb{P}^s \times \mathbb{P}^q$ is given by a series $L + L'$, whose dimension is at least $\dim(L) + 1 = nk + n + k + 1$. It follows that W spans a space of dimension at least $nk + n + k + 1$, a contradiction. Since W passes through $k + 1 \geq 2$ points of X , its image into \mathbb{P}^q can be trivial only when W is given by $k + 1$ components, $W = W_0 \cup \dots \cup W_k$ and each W_i is contained in a fiber of the projection $X \rightarrow \mathbb{P}^q$. By monodromy (Remark 2.14), each W_i has the same dimension n and spans a space of the same dimension q' , in the fiber. Call H_0, \dots, H_k the projections of these spaces to \mathbb{P}^s . Since W' spans a space of dimension $nk + n + k$, then the H_i 's span a space of the same dimension. Thus, if the H_i 's are not linearly independent, then W spans a space of dimension at least $nk + n + k + 1$, by Lemma 3.1, a contradiction. It follows that the span of the H_i 's has dimension $nk + n + k = q'k + q' + k$, so that $q' = n$. This means that each W_i projects to a subspace H_i of dimension n in \mathbb{P}^s . Since each W_i sits in a fiber of $X \rightarrow \mathbb{P}^q$, this implies that each W_i is linear, and these subspaces are independent.

Assume $n' = 0$. Then necessarily W consists of $k + 1$ components $W = W_0 \cup \dots \cup W_k$, each W_i being contained in a fiber of the projection. As above, it turns out that W spans a space of dimension $kq' + q' + k$. Thus $q' > n$ yields a contradiction. Hence $q' = n$, so that every W_i is linear.

It follows, from the previous analysis, that necessarily W is the union of $k + 1$ linearly independent subspaces of dimension n . We get then that either X is not k -defective, so it is k -identifiable by Theorem 2.11, or it is k -defective, and the $(k + 1)$ -contact variety W is the union of $k + 1$ linearly independent subspaces of dimension n . In the latter case, the k -th secant order of W is known to be 1 (see Example 2.2). Thus X is k -identifiable, by Theorem 2.9. \square

Proof of the main Theorem Let k the any positive integer such that $(k + 1)m < 2^{m-1} - 1$, so that the k -secant variety of $(\mathbb{P}^1)^{m-1}$ cannot span $\mathbb{P}^{2^{m-1}-1}$. By the main result of [6], $(\mathbb{P}^1)^{m-1}$ is not k -defective. Then the previous Lemma implies that $(\mathbb{P}^1)^m$ is k -identifiable. \square

Indeed, the previous Lemma also prove the following, stronger statement:

Theorem 3.3. *Let X be a product of $m > 5$ copies of \mathbb{P}^1 , embedded in the projective space \mathbb{P}^r , $r = 2^m - 1$, by the standard Segre embedding. If $k + 1 \leq 2^{m-1}/m$, then X is not k -weakly defective.*

Proof. We know from the proof of the previous Lemma, that if X is k -weakly defective, then a general hyperplane H tangent at $k + 1$ general points P_0, \dots, P_k of X , is tangent along a union of linear spaces. Thus it can only be tangent along fibers of some projection $X \rightarrow \mathbb{P}^1$, because the product does not contain other lines. This is clear for $\mathbb{P}^1 \times \mathbb{P}^1$, while for higher dimensional products one can argue by induction, on some projection $(\mathbb{P}^1)^m \rightarrow (\mathbb{P}^1)^{m-1}$. Thus, by symmetry, X can be k -weakly defective only when a general $(k + 1)$ -tangent hyperplane H is tangent along all the fibers passing through P_0, \dots, P_k . But then, for a general choice of a point Q in some fiber passing through P_0 , a general hyperplane tangent to P_0, P_1, \dots, P_k is also tangent at Q, P_1, \dots, P_k . By the same argument, it is also tangent along any fiber passing through Q . Arguing again in this way, we get that a general H must be tangent (thus must contain) any point of X . An obvious contradiction. \square

4. RESULTS FOR SMALL m

Proposition 4.1. *The product X of 5 copies of \mathbb{P}^1 is not 4-identifiable. Through a general point of $S^5(X)$ one finds exactly two 5-secant, 4-spaces.*

Proof. Indeed, we prove that through 5 general points of X one can find an irreducible elliptic normal curve $C \subset \mathbb{P}^9$, contained in X . Since a general point of the \mathbb{P}^9 , spanned by C , sits in exactly two subspaces of dimension 4, 5-secant to an irreducible elliptic normal curve (by [9] Proposition 5.2), it follows that the 4-th secant order of X is at least 2. In particular, X is 4-weakly defective, by [9], proposition 2.7, and the 4-th contact locus contains an elliptic normal curve as C . A computer aided computation, at 5 specific points of X , proves that indeed the 5-contact locus of X is exactly an irreducible elliptic normal curve of degree 12. The computation has been performed with the Macaulay2 Computer Algebra package [13], with the script described in [4]. Thus 4-th secant order of X is 2 (by Theorem 2.9) and the claim is proved.

To prove the existence of the curve C passing through 5 general points P_0, \dots, P_4 of X , we start with the product of three lines $X' = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Through the 5 points $P'_0, \dots, P'_4 \in X'$, projection of the P_i 's, one can find a 2-dimensional family \mathcal{F} of elliptic normal curves C' of degree 6. Indeed $X' \subset \mathbb{P}^5$ is a sextic threefold with elliptic curve sections, and there is a 2-dimensional family of hyperplanes passing through 5 general points of X . \mathcal{F} is parametrized by points of some plane Π , obtained by projecting \mathbb{P}^5 from the span of the P'_i 's.

Consider now the product X'' of the two remaining copies of \mathbb{P}^1 , so that $X = X' \times X''$. We also get 5 distinguished general points $P''_0, \dots, P''_4 \in X''$. For any curve C' of the family \mathcal{F} , we have a 7-dimensional family of embeddings $C' \rightarrow X''$. Thus, adding the

automorphisms of C' , for $C' \in \mathcal{F}$ general, we may assume that each P'_i , $i = 1, \dots, 4$, goes to the corresponding P''_i . The condition that P'_0 goes to P''_0 determines two algebraic condition on the family, hence two algebraic curves on Π . Thus, there is at least one curve C' of the family, for which P'_0 goes to P''_0 . This determines an elliptic normal curve C in X , passing through the 5 given general points P_i 's. The fact that C is irreducible, for a general choice of the points, follows by the computer-aided computation, on a specific example. \square

For 6 copies of \mathbb{P}^1 , the maximal value for which k -identifiability makes sense is $k_{\max} = 9$. Our result gives that the product X of 6 copies of \mathbb{P}^1 is k -identifiable, for $k = 1, \dots, 5$. The k -identifiability of $(\mathbb{P}^1)^6$, for $k = 6, 7, 8, 9$, can be directly checked by a computer-aided procedure. Indeed, the following observation reduces our problem to check only what happens for the maximal number k such that $mk + m + k < r$.

Proposition 4.2. *Assume that $km + m + k < r$ and X is not k -weakly defective. Then X is not $(k - 1)$ -weakly defective.*

Proof. Fix $k + 1$ general points $P_0, \dots, P_k \in X$. The family of hyperplanes containing the tangent space T_{X, P_1, \dots, P_k} is irreducible, so a general hyperplane tangent to X at P_0, \dots, P_k is the limit of a family of hyperplanes tangent at P_1, \dots, P_k . Since the general element of this last family has a zero dimensional contact locus, the claim follows. \square

Now, by Corollary 2.11, it is enough to compute that some hyperplane tangent to X at some points P_0, \dots, P_k , is in fact tangent only at those $k + 1$ points. Using this procedure with 9 points of $(\mathbb{P}^1)^6$, a computer-aided computation, using the script in [4], proves the following:

Proposition 4.3. *For $m = 6$ and for all $k \leq k_{\max} = 9$, the product X of 6 copies of \mathbb{P}^1 is k -identifiable.*

For sure, with a more advanced technical equipment, one can analyze products with more copies of \mathbb{P}^1 . Nevertheless, Proposition 4.3 already provides an initial evidence for our Conjecture 1.2.

REFERENCES

- [1] H. ABO, G. OTTAVIANI, C. PETERSON. *Induction for secant varieties of Segre varieties*. Trans. Amer. Math. Soc. **361** (2009), 767-792.
- [2] B. ADLANSDVIK. *Joins and Higher secant varieties*. Math. Scand. **61** (1987), 213-222.
- [3] E.S. ALLMAN, C. MATIAS, J.A. RHODES. *Identifiability of parameters in latent structure models with many observed variables*. Ann. Statist. **37** (2009), 3099-3132.
- [4] C. BOCCI, L. CHIANTINI. *Appendix to "On the identifiability of binary Segre products"*. Available at: www.mat.unisi.it/personalpages/chiantini/plapp.pdf.
- [5] M.V. CATALISANO, A.V. GERAMITA, A. GIMIGLIANO. *Ranks of tensors, secant varieties of Segre varieties and fat points*. Linear Algebra Appl. **355** (2002), 263-285.
- [6] M.V. CATALISANO, A.V. GERAMITA, A. GIMIGLIANO. *Secant Varieties of $(\mathbb{P}^1) \times \dots \times (\mathbb{P}^1)$ (n -times) are NOT Defective for $n \geq 5$* . J. Alg. Geom. **20** (2011), 295-327.
- [7] L. CHIANTINI, C. CILIBERTO. *Weakly defective varieties*. Trans. Amer. Math. Soc. **354** (2002), 151-178.
- [8] L. CHIANTINI, C. CILIBERTO. *On the classification of defective threefolds*. in Projective Varieties with Unexpected Properties, Proceedings of the Siena Conference, C. Ciliberto, A. V. Geramita, B. Harbourne, R. M. Mirò-Roig, K. Ranestad ed., W. de Gruyter (2005), 131-176.
- [9] L. CHIANTINI, C. CILIBERTO. *On the concept of k -secant order of a variety*. J. London Math. Soc. **73** (2006), 436-454.

- [10] C. CILIBERTO, M. MELLA, F. RUSSO. *Varieties with one apparent double point*. J. Alg. Geom. **13** (2004), 475–512.
- [11] M. DALE. *Terracini’s lemma and the secant variety of a curve*. Proc. London Math. Soc. **49** (1984), 329–339.
- [12] R. ELMORE, P. HALL, A. NEEMAN. *An application of classical invariant theory to identifiability in non-parametric mixtures*. Ann. Inst. Fourier **55** (2005), 1–28.
- [13] D. GRAYSON, M. STILLMAN. *Macaulay 2, a software system for research in algebraic geometry*. Available at: www.math.uiuc.edu/Macaulay2/.
- [14] J.B. KRUSKAL. *Three-way arrays: rank and uniqueness of trilinear decompositions, with applications to arithmetic complexity and statistics*. Lin. Alg. Applic. **18** (1977), 95–138.
- [15] A. TERRACINI. *Sulle V_k per cui la varietà degli S_h , $(h+1)$ -secanti ha dimensione minore dell’ordinario*. Rend. Circ. Mat. Palermo **31** (1911), 392–396.
- [16] F.L. ZAK. *Tangents and secants of varieties*. AMS Bookstore publications, Transl. Math. Monog. **127** (1993).

CRISTIANO BOCCI, UNIVERSITÀ DEGLI STUDI DI SIENA, DIPARTIMENTO DI SCIENZE
MATEMATICHE E INFORMATICHE, PIAN DEI MANTELLINI, 44, I – 53100 SIENA

E-mail address: bocci24@unisi.it

URL: <http://www.mat.unisi.it/newsito/docente.php?id=16>

LUCA CHIANTINI, UNIVERSITÀ DEGLI STUDI DI SIENA, DIPARTIMENTO DI SCIENZE
MATEMATICHE E INFORMATICHE, PIAN DEI MANTELLINI, 44, I – 53100 SIENA

E-mail address: chiantini@unisi.it

URL: <http://www.mat.unisi.it/newsito/docente.php?id=4>